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Testing Holographic Principle from Logarithmic and Higher Order Corrections to Black Hole Entropy

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ABSTRACT

The holographic principle is tested by examining the logarithmic and higher order corrections to the Bekenstein-Hawking entropy of black holes. For the BTZ black hole, I find some disagreement in the principle for a holography screen at spatial infinity beyond the leading order, but a holography with the screen at the horizon does not, with an appropriate choice of a period parameter, which has been undetermined at the leading order, in Carlip's horizon-CFT approach for black hole entropy in any dimension. Its higher dimensional generalization is considered to see a universality of the parameter choice. The horizon holography from Carlip's is compared with several other realizations of a horizon holography, including induced Wess-Zumino-Witten model approaches and quantum geometry approach, but none of the these agrees with Carlip's, after clarifications of some confusions. Some challenging open questions are listed finally.

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1 Introduction

In the recent ten years, there have been enormous studies on the "holographic principle", which states that the world in the bulk space-time is encoded on its boundary surface: holographic screen [1]. In other words, this principle says, roughly speaking,

$$Z_X \sim Z_{\partial X} \tag{1.1}$$

with the partition functions Z_X on the bulk space-time X and $Z_{\partial X}$ on its boundary ∂X. This has been now well-tested when anti-de Sitter spaces are involved in the bulk, and some appropriate conformal field theories are considered on the infinite boundary in the context of string theory (AdS/CFT) [2]. Similar correspondences for de-Sitter [3, 4, 5] and flat spaces [4, 6] have been considered recently, but concrete realizations have not been achieved yet.

On the other hand, when there are black holes in the bulk space-time X, it seems that, according to 't Hooft and Susskind [1], collection of event horizons can be considered as holographic screens also. Similarly, when there is the cosmological horizon, this can then be a holographic screen also [7]. There is a strong evidence for this in the computation of the Bekenstein-Hawking(BH) entropy from a conformal field theory living on the *horizon* [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Since the BH entropy is the exponent of the partition functions at the "leading order", this result implies the relation (1.1) holds at the leading order. However, this is in contrast to Strominger's computation on the BTZ black hole entropy from a conformal field theory at *spatial infinity* [18] that implies a holography (1.1) at spatial infinity *at the same (leading) order*. So, at least for the leading order, both the holography at spatial infinity and that of the horizon give an identical result, and one can not distinguish between these very different approaches.

In this paper, I show that, for the BTZ black hole, the two holography prescriptions give different results, so these can be differentiated beyond the leading order. I find that, remarkably, there is some disagreement in the holography at spatial infinity beyond the leading order. This may be in contrast to the AdS/CFT predictions. On the other hand, the holography at the horizon may be satisfied by choosing an appropriate period P, which has been arbitrary for the leading-order computations, of the metric diffeomorphism parameter ξ^μ in Carlip's horizon-CFT approach of black hole entropy in any dimension [12, 13, 14]. The higher dimensional generalization is considered from the Schwarzschild-anti-de Sitter black holes in $d \geq 4$ to see a universality of the parameter choice. This horizon holography from Carlip's is compared with several other realizations of a horizon holography, including induced Wess-Zumino-Witten model approaches for the Lorenzian and the Euclidean BTZ black holes; quantum geometry approach for the four-dimensional Schwarzschild and Schwarzschild-AdS black holes. But, I find

that none of these other approaches agrees with Carlip's, after clarifications of some confusions. Some challenging open questions are listed finally.

2 General Logarithmic Corrections to the Bekenstein-Hawking entropy

I start by considering the density of states $\Omega(E)$ in the *micro-canonical* ensemble, where the total energy E is fixed and always bigger enough than the *tolerance* δE, i.e., $\delta E \ll E$,

$$\begin{aligned} \Omega(E) &= \sum_r \delta(E_r - E)\delta E & (2.1)\\ &= \frac{\delta E}{2\pi}\int_{-\infty}^{\infty} d\beta' e^{\hat{\beta}E} Z[\hat{\beta}], & (2.2)\end{aligned}$$

where $\hat{\beta} = \beta + i\beta'$, and β is an arbitrary parameter which can be chosen at will. The partition function $Z[\hat{\beta}]$ in the *canonical* ensemble for the complex $\hat{\beta}$ is defined as

$$Z[\hat{\beta}] = \sum_r e^{-\hat{\beta}E_r}. \tag{2.3}$$

On the other hand, from the Laplace transformation for $\Omega(E)$ of (2.1), this can be alternatively expressed as

$$Z[\hat{\beta}] = \int_{-\infty}^{\infty} \frac{dE}{\delta E}\Omega(E)e^{-\hat{\beta}E}. \tag{2.4}$$

However, note that the existence of the canonical partition function depends on the convergence of the Laplace transform (2.4): When the Laplace transform does not converge as occurs in the Schwarzschild black hole, where the heat capacity is negative such as it is thermodynamically unstable, one must return to the original definition (2.1) to compute $\Omega(E)$ [19, 20]; but, for a thermodynamically stable system, one can always use the formula (2.2). Moreover, in this computation it is important to note that *only the $\beta' \leftrightarrow -\beta'$ symmetric terms in the integrand $e^{\hat{\beta}E}Z[\hat{\beta}]$ of (2.2) are relevant by construction*; this property has a crucial role, as can be seen in the later part, when one computes (2.2) perturbatively beyond the first correction.

In the stationary phase method, the integral (2.2) can be perturbatively evaluated around $\beta' = 0$, where the usual energy formula at the equilibrium temperature $T = \beta^{-1}$

$$E = -\left.\frac{\partial \mathrm{ln} Z[\hat{\beta}]}{\partial \hat{\beta}}\right|_{\beta'=0} \tag{2.5}$$

is satisfied, as follows

$$\Omega(E) = e^{\beta E}Z[\beta] \times \frac{\delta E}{2\pi}\int_{-\infty}^{\infty} d\beta' \mathrm{exp}\left\{-\frac{1}{2}B_2\beta'^2 + \sum_{n\geq 3}^{\infty}\frac{1}{n!}B_n(i\beta')^n\right\}, \tag{2.6}$$

where

$$B_n = \frac{\partial^n \ln Z[\hat{\beta}]}{\partial \hat{\beta}^n}\bigg|_{\beta'=0}. \tag{2.7}$$

It is a standard result [21] that the (micro-canonical) entropy $S = \ln\Omega(E)$ is computed as

$$S = S_c - \frac{1}{2}\ln\left[\frac{2\pi C_x T^2}{(\delta E)^2}\right] + (\text{higher order terms}), \tag{2.8}$$

where the second term comes from the Gaussian integral $\int_{-\infty}^{\infty} d\beta' \exp(-B_2\beta'^2/2) = \sqrt{2\pi/B_2}$ and $B_2 = C_x T^2$ (C_x is the specific heat with a fixed extensive parameter x). And

$$S_c = \beta E + \ln Z[\beta] \tag{2.9}$$

is the entropy defined in the canonical ensemble; I shall mean 'entropy' as the micro-canonical entropy later on unless it is stated otherwise. The 'higher order terms' represents the correction terms from the integral $\int_{-\infty}^{\infty} d\beta' \exp[\sum_{n\geq 3}^{\infty} B_n (i\beta')^n/n!]$. In the usual statistical mechanics, T is independent on the system size, i.e., intensive variable, such as T^2 term in the logarithmic term of (2.8) can be neglected in the thermodynamic limit since C_x is an extensive quantity, which can be arbitrarily large like as E and S [2]. But still, the term of tolerance δE is neglected since δE is defined to be smaller enough than any available energy fluctuation, i.e., $\delta E \ll \Delta E = C_x T^2$ for the mean-square fluctuation in the energy [22].

However, in the application of this statistical mechanical formulation into black hole thermodynamics by the Euclidean path-integral approach [23, 24] T depends on the system size, i.e., the horizon size, such as the T^2 term can not be neglected anymore in the logarithmic term of (2.8). This is one of the big differences between the usual thermodynamics and black hole thermodynamics. But one can still assume safely $\delta E \ll$ *any available energy fluctuation*, by considering *large* black holes. Moreover, the relation (2.8) between the micro-canonical entropy and the canonical entropy S_c shows a perturbative correction [25] to the BH entropy [3] [23, 24]

$$S_c \equiv S_{BH} = \frac{A}{4\hbar G}, \tag{2.10}$$

where A is the horizon area, and G is the Newton's constant; when there is rotational degrees of freedom, one can generalize straightforwardly to the grand-canonical partition function and its associated density of states, but this will not be considered in this paper.

[2] I thank K. Huang for a helpful comment on this.

[3] If I include quantum corrections, I need to generalize the Einstein gravity to a higher curvature gravity with a *possible* UV-finite lnA correction. But this would not affect our result for large black holes. See discussion 1 in section 8 for details.

3 General Logarithmic Corrections to the BH entropy from the Cardy Formula

There is an analogous situation in the two-dimensional conformal field theory on a two-torus, where one can perturbatively evaluate the density of states. To see this, I begin with the partition function of the conformal field theory on a torus [26, 27][4]

$$Z[\tau,\bar{\tau}] = Tre^{2\pi i\tau(L_0-\frac{c}{24})}e^{-2\pi i\bar{\tau}(\bar{L}_0-\frac{\bar{c}}{24})} \tag{3.1}$$

with the modular parameters $\tau, \bar{\tau}$ and the Virasoro generators $L_m, \bar{L}_m$ on the "plane" with central charges $c, \bar{c}$, with the algebras in the standard form,

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \\
&[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} + \frac{\bar{c}}{12}m(m^2-1)\delta_{m+n,0}, \\
&[L_m, \bar{L}_n] = 0.
\end{aligned} \tag{3.2}$$

The density of states $\rho(\Delta,\bar{\Delta})$ for the eigenvalues $L_0 = \Delta, \bar{L}_0 = \bar{\Delta}$ is given as a contour integral (I suppress the $\bar{\tau}$-dependence for simplicity, but the computation is similar to the τ-part)

$$\rho(\Delta) = \int_C d\tau\ e^{-2\pi i(\Delta-\frac{c}{24})\tau} Z[\tau], \tag{3.3}$$

where the contour C encircles the origin in the complex $q = e^{2\pi i\tau}$ plane [9], and the tolerance is given by $\delta\Delta = 1$ by definition. The general evaluation of this integral would be impossible unless $Z[\tau]$ is known completely. But, due to the modular invariance of (3.1), one can easily compute its asymptotic formula through the steepest descent approximation. In particular, (3.1) is invariant under $\tau \to -1/\tau$ [26] such that

$$Z[\tau] = Z[-1/\tau] = e^{-2\pi i(\Delta_{\min}-\frac{c}{24})\tau}\tilde{Z}[-1/\tau], \tag{3.4}$$

where $\tilde{Z}[-1/\tau] = Tre^{-2\pi i(L_0-\Delta_{\min})/\tau}$ approaches a constant value $\rho(\Delta_{\min})$ as $\tau \to i0_+$, which defines the steepest descent path for a "real" value of $\Delta \geq \Delta_{\min}$. With the help of this property, (3.3) is evaluated as, by expanding the integrand around the steepest descent path τ_*,

$$\begin{aligned}
\rho(\Delta) &= \int_C d\tau\ e^{\eta(\tau)}\tilde{Z}[1/\tau] &(3.5)\\
&= e^{\eta(\tau_*)}\tilde{Z}[-1/\tau^*] \times \int_C d\tau\ \exp\left\{\frac{1}{2}\eta^{(2)}(\tau-\tau_*)^2 + \sum_{n\geq 3}^{\infty}\frac{1}{n!}\eta^{(n)}(\tau-\tau_*)^n\right\} \\
&\quad \times \left[1 + \sum_{m\geq 1}^{\infty}\frac{1}{m!}\tilde{Z}^{-1}\tilde{Z}^{(m)}(\tau-\tau_*)^m\right], &(3.6)
\end{aligned}$$

[4]Compare to the derivation of Carlip [27], who used a slightly different partition function $\hat{Z}[\tau,\bar{\tau}] = Tre^{2\pi i\tau L_0}e^{-2\pi i\bar{\tau}\bar{L}_0}$, which is *not* modular invariant. But, the result for the density of states $\rho(\Delta)$ is the same.

where $\eta(\tau) = -2\pi i\Delta_{\text{eff}}\tau + 2\pi i c_{\text{eff}}/(24\tau)$, which dominates $\tilde{Z}[1/\tau]$ in the region of interest, gets the maximum

$$\eta(\tau_*) \;=\; 2\pi\sqrt{\frac{c_{\text{eff}}\Delta_{\text{eff}}}{6}} \tag{3.7}$$

with

$$\tau_* \;=\; i\sqrt{\frac{c_{\text{eff}}}{24\Delta_{\text{eff}}}} \tag{3.8}$$

when

$$\Delta_{\text{eff}} \gg \frac{c_{\text{eff}}}{24} \tag{3.9}$$

is satisfied. Here, $\eta^{(n)} = (d^n\eta/d\tau^n)|_{\tau=\tau_*}$, $\tilde{Z}^{(m)} = (d^n\tilde{Z}/d\tau^n)|_{\tau=\tau_*}$, and $c_{\text{eff}} = c - 24\Delta_{\min}$, $\Delta_{\text{eff}} = \Delta - c/24$; $\Delta_{\min}$ is the minimum of Δ. Here, I am assuming "$c_{\text{eff}}, \Delta_{\text{eff}} > 0$" since, otherwise, the saddle point approximation is not valid for *real* valued c_{eff}, Δ_{eff}. Then, in the limit of $\epsilon \to \infty$ with $\tau_* = i/\epsilon$, the higher order correction terms in the bracket of (3.6) are exponentially suppressed as $e^{-2\pi\epsilon(\Delta-\Delta_{\min})}$, hence (3.6) is simplified as, up to the exponentially suppressing terms,

$$\rho(\Delta) = e^{2\pi\sqrt{c_{\text{eff}}\Delta_{\text{eff}}/6}} \times \int_C d\tau\, \exp\left\{\frac{1}{2}\eta^{(2)}(\tau-\tau_*)^2 + \sum_{n\geq 3}^{\infty}\frac{1}{n!}\eta^{(n)}(\tau-\tau_*)^n\right\}, \tag{3.10}$$

where I have used $\tilde{Z}[i\infty] = 1$. This is known as the Cardy formula [26]. Note that here I need $c_{\text{eff}}\Delta_{\text{eff}} \gg 1$ in order that the approximation is reliable, i.e., $e^{\eta(\tau_*)}$ dominates in the integral of (3.5), as well as the condition (3.9) such as $\tilde{Z}[-1/\tau]$ is slowly varying near τ_*. Then, the entropy $S_{CFT} = \ln\,\rho(\Delta)$ of this τ sector becomes [27]

$$S_{CFT} = S_0 + \frac{1}{4}\ln\left[\frac{c_{\text{eff}}}{96\Delta_{\text{eff}}^3}\right] + (\text{h.o.t.}), \tag{3.11}$$

where

$$S_0 = 2\pi\sqrt{c_{\text{eff}}\Delta_{\text{eff}}/6}, \tag{3.12}$$

the second term comes from the contour integral $\int_C d\tau\, \exp(\eta^{(2)}(\tau-\tau_*)^2/2) = \sqrt{2\pi/|\eta^{(2)}|}$, and '(h.o.t)' denotes the higher order terms from $\int_C d\tau\, \exp[\sum_{n\geq 3}^{\infty}\eta^{(n)}(\tau-\tau_*)^n/n!]$. Now, by recovering $\bar{\tau}$-part also, the total entropy becomes

$$S_{CFT^2} = S_0 + \bar{S}_0 + \frac{1}{4}\ln\left[\frac{c_{\text{eff}}\bar{c}_{\text{eff}}}{(96)^2\Delta_{\text{eff}}^3\bar{\Delta}_{\text{eff}}^3}\right] + (\text{h.o.t.}). \tag{3.13}$$

This shows a similar logarithmic correction term as the first-order correction in the bulk (2.8). But, it seems in a priori that these CFT entropies S_{CFT,CFT^2} need not be related to the bulk entropy S, which has been computed in a completely different context. However, in the context of the holographic principle this "might" be related if one considers the conformal field theory in the above computation lives on a holographic screen. This remarkable connection does exit at the leading order, as has been first shown by Strominger [18]; he has shown that the CFT entropy associated with the asymptotic isometry $SO(2,2)$ at spatial infinity of the BTZ black hole agrees with the BH entropy, which is the entropy in the bulk at the leading order. This has been considered as a concrete example of the $\mathrm{AdS}_3/\mathrm{CFT}_2$. Moreover, a similar coincidence has been observed in the $\mathrm{dS}_3/\mathrm{CFT}_2$ context also, where there is the cosmological horizon instead, though its associated Cardy formula has not been proved yet [3][5], and though it has a conceptual problem of the meaning of the holographic screen beyond the casually connected region.

On the other hand, recently it has been shown that there is also a conformal filed theory on the event horizon for arbitrary black holes in any dimension, and its associated CFT entropy agrees with the BH entropy at the leading order [12, 15, 16, 17]. This latter approach is conceptually better than the former one of Strominger for the following reasons. First, in the former case, as noted by Carlip [12, 16], one can *not* distinguish the BTZ black hole with a point "star" with some appropriate mass and spin sitting at the origin of AdS space [29]; this is in contrast to the holographic principle, where the data on a screen should distinguish the black holes from the point star by definition; however, this is automatically satisfied for the latter approach by construction. Second, in de-Sitter case, there is an additional conceptual problem that the screen is located in the casually-disconnected region as I mentioned above. However, this problem does not appear in the latter approach either. But, unfortunately one can not distinguish those two very different holography schemes at the leading-order computations. This raises the following question of the higher order corrections: *Can we determine the correct one by assuming the holographic principle $Z_X \sim Z_{\partial X}$ even at the higher orders ?*

4 Test I: The BTZ Black Hole

a In the Bulk:

The BTZ black hole solution [30] is the solution to the vacuum Einstein equation in 2+1 dimensions with a negative cosmological constant $\Lambda = -1/l^2$. The metric is given by

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2, \tag{4.1}$$

[5] This has been proved only recently by the author [28]. I will discuss on this later.

where

$$N^2 = -8GM + \frac{r^2}{l^2} + \frac{16G^2J^2}{r^2},\ \ N^\phi = -\frac{4GJ}{r^2}. \tag{4.2}$$

M and J are the ADM mass and angular momentum of the black hole ($M > 0, |J| \leq ML$), and there are inner and outer horizons at

$$r_\pm = 2\sqrt{G}l\sqrt{M \pm \sqrt{M^2 - (J/l)^2}}. \tag{4.3}$$

The BH entropy, the Hawking temperature, and the angular velocity of the horizon are given by, respectively,

$$\begin{aligned} S_{BH} &= \frac{2\pi r_+}{4G}, && (4.4)\\ T &= \frac{\kappa}{2\pi} = \frac{r_+^2 - r_-^2}{2\pi l^2 r_+}, && (4.5)\\ \Omega &= \frac{4GJ}{r_+^2}. && (4.6) \end{aligned}$$

The heat capacity $C_J = (\partial M/\partial T)_J$ for 'J=fixed' becomes, after some computation,

$$C_J = \frac{4\pi r_+(r_+^2 - r_-^2)}{8G(r_+^2 + 3r_-^2)}. \tag{4.7}$$

Note that C_J is always positive since $r_+ > r_-$ for non-extremal black holes ($T > 0$) such as the canonical (T =fixed) or grand-canonical ($T = \Omega$ =fixed) ensemble exists without considering the equilibrium of black holes with a *hypothetical enveloping-box*, in contrast to the black hole solutions in asymptotically flat space [31]. But, since I am interested in the *large* black holes of $r_+ \gg l$, as appropriate in our perturbative computation, the canonical ensemble, where Ω is negligible, is enough. Then from

$$\begin{aligned} C_J T^2 &= \frac{(r_+^2 - r_-^2)^3}{8\pi G l^4 r_+ (r_+^2 + 3r_-^2)} \\ &\approx \frac{(8G)^2}{4^3\pi^4 l^4} S_{BH}^3 && (4.8) \end{aligned}$$

in the $r_+ \gg l$ regime, one can easily find the entropy in the bulk as [6], from (2.8),

$$\begin{aligned} S &= S_{BH} - \frac{1}{2}\ln\left[\frac{S_{BH}^3 (8G)^2}{32\pi^3 l^4 (\delta E)^2}\right] + (\text{h.o.t.}) \\ &= S_{BH} - \frac{3}{2}\ln S_{BH} + \frac{1}{2}\ln[32\pi^3 l^4 (\delta E)^2/(8G)^2] + (\text{h.o.t.}). && (4.9) \end{aligned}$$

[6]The "$\ln S_{BH}$" corrections were known earlier in the one-loop corrections due to quantum fields on the black hole background in Refs.[32, 33, 34] (I thank R. B. Mann for informing this); on the contrary, our corrections are essentially due to the fluctuations of the black hole metric itself even without the matters. However, the coeffients are not universal, but depend on the matter contents. Furthermore, its connection to the CFT's logarithmic correction was also studied [35] in the context of string/black-hole correspondence [36], where the string's ends are frozen on the horizon.

b At Spatial Infinity:

Now, let us compare with the CFT computation from Cardy's formula on a boundary. In this subsection first, let us consider the screen at spatial infinity as in the usual AdS/CFT. To this end, I note that the associated Virasoro algebra has two independent sectors as in (3.2), which have eigenvalues of $L_0, \bar{L}_0$ and central charges [37, 18, 38, 3, 14] as

$$\begin{aligned} \Delta_{\text{eff}}, \bar{\Delta}_{\text{eff}} &= \frac{(r_+ \pm r_-)^2}{16Gl} \\ &\approx \frac{8G}{32\pi^2 l}\, S_{BH}^2, \end{aligned} \tag{4.10}$$

$$c = \bar{c} = \frac{12l}{8G}, \tag{4.11}$$

where $r_+ \gg l$ is considered in (4.10). Then, one can easily find the entropy for the CFT, from (3.13), as

$$S_{CFT^2} = S_{BH} - 3\ \text{ln} S_{BH} + \text{ln}[64\pi^3 l^2/(8G)^2] + (\text{h.o.t.}), \tag{4.12}$$

where I have used

$$\begin{aligned} S_0 + \bar{S}_0 &= 2\pi\sqrt{c_{\text{eff}}\Delta_{\text{eff}}/6} + 2\pi\sqrt{\bar{c}_{\text{eff}}\bar{\Delta}_{\text{eff}}/6} \\ &= S_{BH} \end{aligned} \tag{4.13}$$

with a choice $\Delta_{\min} = 0$ [18] [7]. This clearly shows the factor of 2 mismatch, with respects to the bulk result (4.9), at the logarithmic order for a large black hole, i.e., large S_{BH}.[8] Hence, one finds that:

There is some disagreement in the holographic correspondence between 3 dimensional gravity and 2 dimensional CFT, associated with the asymptotic isometry group $SO(2,2)$ at the spatial infinity, beyond the leading order.

Although there is no higher dimensional analogy [42], this higher order disagreement of the holographic principle in three dimensions might be far-reaching consequence because of its frequent appearance in the many higher dimensional black holes in string theory [43] as a sub-sector.

[7] This vacuum is crucial in obtaining the correct BH entropy even when a scalar field is coupled. See Ref.[39].

[8] A similar mismatch has been noted, for the first time, by Carlip [40] in a different context of quantum geometry. But, as will be explained in later sections, his observation was due to some misunderstanding of the entropy in quantum geometry [41]. Moreover, he has not considered its implication to the holographic principle since the quantum geometry approach also concerns about the "horizon" states, and the associated "bulk" entropy is not considered.

c At the Horizon:

At the horizon r_+, the associated CFT has several different features compared to that for spatial infinity. First of all, there is only "one" copy of the Virasoro algebra instead of the two copies (3.2) at spatial infinity, though its origin is at the "classical" level like as in spatial infinity [12]; the intuitive understanding of this is not clear to us, but this has been observed in various different contexts [15, 16, 17] also. Second, the Virasoro algebra at the horizon is quite universal for arbitrary black holes and any dimension (except the extremal black holes and two-dimensional black holes), in contrast to the Virasoro algebra for AdS_3 space at spatial infinity [9]. Though the "most" general from of the proper boundary conditions near the horizon is not clear, for a "quite" general form of the boundary conditions near the horizon and an appropriate choice of $\Delta_{\rm min}$ ($\Delta_{\rm min} = 0$ in our case), one obtains a Virasoro algebra [12, 40] [10]

$$\begin{aligned} c_{\rm eff} &= \frac{3A}{2\pi G}\frac{2\pi}{\kappa P}, \\ \Delta_{\rm eff} &= \frac{A}{16\pi G}\frac{\kappa P}{2\pi}, \end{aligned} \tag{4.14}$$

where A is the horizon area, κ is the surface gravity, and P is the periodicity of the diffeomorphism parameter ξ^t. Here, note that "$\kappa P/(2\pi)$" terms of (4.14) cancel in the computation of the leading term of the CFT entropy (3.12)

$$S_0 = 2\pi\sqrt{c_{\rm eff}\Delta_{\rm eff}/6} = \frac{A}{4G}, \tag{4.15}$$

which is the BH entropy, such as P is *not* determined at the leading order.

However, an explicit P-dependent term appears for higher order corrections as follows, from (3.11),

$$S_{CFT} = S_{BH} - \frac{1}{2}{\rm ln}S_{BH} - {\rm ln}(\kappa P) + \frac{1}{2}{\rm ln}(8\pi^3) + ({\rm h.o.t.}). \tag{4.16}$$

Now, by using

$$\begin{aligned} \kappa &= \frac{r_+^2 - r_-^2}{l^2 r_+} \\ &\approx \frac{8G}{4\pi l^2}S_{BH} \end{aligned} \tag{4.17}$$

[9] There seems to be a very strong hint on the universality of AdS_3 Virasoro algebra for AdS_d also. For a recent achievement in the special context of string theory and pp-waves see Ref.[44]. But there has been no general proof for more general contexts yet.

[10] There are some mismatches in $c_{\rm eff}$ and $\Delta_{\rm eff}$ with other alternative approaches [16, 17]. But, presumably similar logarithmic corrections would be obtained also if the vacuum is chosen properly [45].

for the large black holes of $r_+ \gg l$ (with a finite J), one finally obtains

$$S_{CFT} = S_{BH} - \frac{3}{2}\ln S_{BH} - \ln P + \frac{1}{2}\ln(96\pi^5 l^4/(8G)^2) + (\text{h.o.t.}). \tag{4.18}$$

Hence, one finds that the bulk entropy S of (4.9) matches with its boundary entropy at the horizon S_{CFT} if one choose P as

$$P \approx 8\pi/\delta E, \tag{4.19}$$

which is a universal constant, independent of A. It is interesting to note that $c_{\rm eff}$ becomes a universal constant also

$$c_{\rm eff} = c \approx \frac{3l^2\delta E}{4G}, \tag{4.20}$$

while $\Delta_{\rm eff} \approx (2G/\delta E l^2\pi^2)S_{BH}^2$, with a choice $\Delta_{\rm min} = 0$.

5 Test II : Schwarzschild-Anti-de Sitter black holes (d ≥ 4)

As the higher dimensional generalization of the analysis of the previous section, let us consider a higher dimensional Schwarzschild black hole solution with the (negative) cosmological constant $\Lambda = -(d-1)(d-2)/2l^2$ (Schwarzschild-AdS). The metric is given by

$$ds^2 = -N^2 dt^2 + N^{-2}dr^2 + r^2 d\Omega_{d-2}^2, \tag{5.1}$$

where

$$N^2 = 1 - \frac{16\pi G M}{(d-2)\Omega_{d-2} r^{d-3}} + \frac{r^2}{l^2}. \tag{5.2}$$

M is the mass of the black holes ($M > 0$), and $d\Omega_{d-2}^2$ is the line element on the unit sphere S^{d-2} [31, 46, 47], and the horizon radius increases as the mass M as follows

$$\frac{16\pi G M}{(d-2)\Omega_{d-2}} = r_+^{d-3}\left(1 + \frac{r_+^2}{l^2}\right), \tag{5.3}$$

where Ω_{d-2} is the area of the unit S^{d-2}: $\Omega_{d-2} = 2\pi^{(d-1)/2}/\Gamma((d-1)/2)$. The BH entropy, the Hawking temperature, and the heat capacity of the horizon are given by, respectively,

$$S_{BH} = \frac{\Omega_{d-2} r_+^{d-2}}{4G}, \tag{5.4}$$

$$T = \frac{1}{4\pi r_+}\left(\frac{d-1}{l^2}r_+^2 + (d-3)\right), \tag{5.5}$$

$$C_J = \frac{(d-2)\Omega_{d-2} r_+^{d-2}}{4G}\left(\frac{(d-1)r_+^2/l^2 + (d-3)}{(d-1)r_+^2/l^2 - (d-3)}\right). \tag{5.6}$$

Note that C_J is *not* always positive, but becomes negative for the small black hole of $r_+ < \sqrt{(d-3)/(d-1)}l$. However, this does not matter in our case since I am only interested in the large black hole of $r_+ \gg l$ such that our perturbative expansion of black hole entropy is meaningful. Then from

$$\begin{aligned} C_J T^2 &= \frac{(d-2)\Omega_{d-2} r_+^{d-4}}{4G(4\pi)^2} \frac{[(d-1)r_+^2/l^2 + (d-3)]^3}{[(d-1)r_+^2/l^2 - (d-3)]} \\ &\approx \frac{(d-2)(d-1)^2}{(4\pi)^2 l^4} \left(\frac{4G}{\Omega_{d-2}}\right)^{2/(d-2)} S_{BH}^{d/(d-2)} \end{aligned} \tag{5.7}$$

for the large black hole, one can easily find the entropy in the bulk [25], from (2.8),

$$S = S_{BH} - \frac{d}{2(d-2)} \ln S_{BH} + \frac{1}{2} \ln \left[\frac{8\pi l^4 (\delta E)^2}{(d-2)(d-1)^2} \left(\frac{\Omega_{d-2}}{4G}\right)^{2/(d-2)} \right] + (\text{h.o.t.}). \tag{5.8}$$

Now, since there is no CFT analogue at spatial infinity, as far as I know, let us consider only the CFT at the horizon. There, the formulas of the eigenvalues $L_0, \bar{L}_0$ and the central charges (4.14) are valid for arbitrary higher dimensions, such as one has the same "formal" higher order corrections as (4.16) also. But now, by using

$$\begin{aligned} \kappa = 2\pi T &= \frac{1}{2r_+} \left(\frac{d-1}{l^2} r_+^2 + (d-3) \right) \\ &\approx \frac{(d-1)}{2l^2} \left(\frac{4G}{\Omega_{d-2}}\right)^{1/(d-2)} S_{BH}^{1/(d-2)} \end{aligned} \tag{5.9}$$

for the large black hole, one can finally obtains

$$S = S_{BH} - \frac{d}{2(d-2)} \ln S_{BH} - \ln P + \frac{1}{2} \ln \left[\frac{32\pi^3 l^4 (\delta E)^2}{(d-1)^2} \left(\frac{\Omega_{d-2}}{4G}\right)^{2/(d-2)} \right] + (\text{h.o.t.}). \tag{5.10}$$

Hence, one finds that the bulk entropy S matches with its associated CFT entropy S_{CFT} if one choose P as

$$P \approx \sqrt{d-2}\; 8\pi/\delta E, \tag{5.11}$$

which is a universal constant, independent of A, as in the three-dimensional case. But, in contrast to the three-dimensional case, c_{eff} does not become a universal constant anymore,

$$c_{\text{eff}} = c \approx \frac{3l^2 \delta E}{\pi(d-1)\sqrt{d-2}} \left(\frac{\Omega_{d-2}}{4G}\right)^{1/(d-2)} S_{BH}^{(d-3)/(d-2)}. \tag{5.12}$$

When the above results for the higher dimensional case of $d \geq 4$ are extrapolated to that of $d = 3$, with an appropriate change of definition of mass $M - \frac{1}{8G} \to M$, the formula (5.12),

which becomes (4.20) exactly, clearly shows that c becomes a universal constant only for $d=3$ accidentally; this is in contrast to the speculation in Ref. [40]. (The reason will be explained in section 7.b.)

For the comparison with the $d=3$ case, the results for $d=4$ ($\Omega_2=4\pi$) are

$$\begin{aligned} S_{CFT} &= S_{BH}-\ln S_{BH}+\frac{1}{2}\ln[4\pi^2 l^4(\delta E)^2/(9G)], \\ c_{\rm eff} &= c\approx \frac{l^2\delta E}{\sqrt{2\pi G}}S_{BH}^{1/2}, \\ P &\approx 8\sqrt{2}\pi/\delta E. \end{aligned} \tag{5.13}$$

6 Beyond the Logarithmic Corrections

So far I have considered the logarithmic-order, including the constant term as well, corrections, which are the first-order corrections to the BH entropy. There I showed that the holographic correspondence with the holographic screen at spatial infinity for the BTZ black hole has some disagreement. However I showed that this mismatch can be resolved by considering a horizon holography with an appropriate choice of the period parameter P. Now, the important question would be of its *consistency*: In the CFT manipulation at the horizon there is only "one" undetermined parameter P at the leading order, but this was fixed by the requirement of the (horizon) holographic principle even with the logarithmic-order corrections. Now then, computing the higher order corrections beyond the logarithmic order would be an important test of the consistency of the holographic principle. So, the question is *whether the P determined at the logarithmic order has the perturbative corrections, or that is enough to get a consistent (horizon) holography, even with all the higher order corrections.*

To this end, let us first consider higher order terms in the density of states $\Omega(E)$ of (2.6) in the bulk. The higher order terms in the sum beyond $n=2$, which is the logarithmic-order that I have studied so far, start from $n=3$, but this term seems to be problematic since its contribution becomes *imaginary* due to $(i\beta')^3$ term. Actually, the imaginary terms always appear for odd $n(\geq 3)$. But, due to the reflection symmetry under $\beta'\leftrightarrow-\beta'$ of the relevant integrand of (2.2), as I have noted earlier (below (2.4)), these terms do not occur in our higher order computations. So, the truly relevant terms start from $n=4$, and these give the higher order corrections beyond the logarithmic order ($n=2$). If I take into account these new corrections, one can evaluate the integral of (2.6) "formally" as follows

$$\Omega(E)=e^{\beta E}Z[\beta]\times\frac{\delta E}{\pi\sqrt{B_2}}x^{1/2}e^{x}K_{1/4}(x)+(\text{h.o.t}), \tag{6.1}$$

such as entropy $S = \ln\Omega(E)$ becomes

$$
\begin{aligned}
S &= S_c + \ln\left[\frac{\delta E}{\pi\sqrt{B_2}} x^{1/2} e^x K_{1/4}(x)\right] + (\text{h.o.t}) \\
&= S_c - \frac{1}{2}\ln\left[\frac{2\pi B_2}{(\delta E)^2}\right] + \frac{B_4}{8(B_2)^2} + O\left(\frac{(B_4)^2}{(B_2)^4}\right),
\end{aligned}
\tag{6.2}
$$

where I have used the following formula in the first line

$$
\int_{-\infty}^{\infty} d\beta e^{-a\beta^2 + b\beta^4} = \sqrt{\frac{2}{a}} x^{1/2} e^x K_{1/4}(x) \tag{6.3}
$$

with the modified Bessel function of the second kind $K_n(x)$ for $x = -a^2/(8b)$; in the second line, I have used the asymptotic series expansion for a large value of $x = -3(B_2)^2/(4B_4)$

$$
K_{1/4}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}\left(1 - \frac{3}{32}\frac{1}{x} + O\left(\frac{1}{x^2}\right)\right). \tag{6.4}
$$

The (h.o.t)'s in (6.1) and (6.2) represent the corrections from $\int_{-\infty}^{\infty} d\beta' \exp[\sum_{n\geq 5}^{\infty} B_n (i\beta')^n/n!]$ and its logarithm, respectively. This asymptotic series expansion might not be convergent for negative values of x, (i.e., $b < 0$), which seems to be a generic feature of black holes in AdS space as in this paper, due to an essential singularity at $x = \infty$. But, since [11], the combination $x^{1/2} e^x K_{1/4}(x)$, which appears in (6.3), has no essential singularity at $x = \infty$–actually there is no singularity at all–, the infinite series of (6.4) may be convergent even for the $b < 0$ case, as well as the $b > 0$ case, by Fuch's theorem; the convergence of that combination for the $b < 0$ case implies an appropriate regularization prescription is required[12]to get the finite answer from the integral (6.3), which is divergent *naively*; otherwise, the canonical ensemble is unstable *again* for black holes in AdS space if one considers the higher order corrections, even though it is stable for the first-order correction, but this does not seem to occur [31, 48]. One might also like to continue this perturbative computation to arbitrary higher orders, but this does not seems to be straightforward since we do not know the integral formula for the integrand $e^{\sum_{n=0}^{N} a_n \beta^n}$ for $N > 4$ in (2.6), which generalizes the formula of (6.3); (6.2) is the best correction as much as we can at present.

On the other hand, in the CFT side, one needs to compute $\rho(\Delta)$ of (3.6) by taking into account the higher order terms in the sum which start from $n \geq 3$ similarly to $\Omega(E)$ computation; here also, one can safely neglect the higher order correction terms in the bracket of (3.6), which is exponentially suppressed as $\tau_* \to i0_+$. This higher order corrections may be evaluated by

[11]The combination $W = x^{1/2} e^x K_n(x)$ satisfies the differential equation, $x^2 W'' + 2x^2 W' - (n^2 - 1/4)W = 0$, while $K_n(x)$ satisfies the modified Bessel's equation $x^2 K_n'' + x K_n' - (x^2 + n^2) K_n = 0$.

[12]For example, $\lim_{x\to\infty} \int_{-x}^{x} d\beta$ instead of the integral of (6.3)

the steepest descent method, which would produce a similar form as (6.2). But, surprisingly an exact, convergent expansion, due to Rademacher [49], exits, and I would like to use this elegant formula for our purpose. Then, the formula reads [50, 51], up to the exponentially suppressed terms, which have been neglected also in the explicit steepest computation,

$$\rho(\Delta) = e^{2\pi\sqrt{c_{\rm eff}\Delta_{\rm eff}/6}} \times \left(\frac{c_{\rm eff}}{96\Delta_{\rm eff}^3}\right)^{1/4} I_1(2\pi\sqrt{c_{\rm eff}\Delta_{\rm eff}/6}) + ({\rm h.o.t.}) \ , \tag{6.5}$$

such as

$$\begin{aligned} S_{CFT} &= S_0 + \ln\left[\left(\frac{c_{\rm eff}}{96\Delta_{\rm eff}^3}\right)^{1/4} I_1(S_0)\right] + ({\rm h.o.t.}) \\ &= S_0 + \ln\left(\frac{c_{\rm eff}}{96\Delta_{\rm eff}^3}\right)^{1/4} - \frac{3}{8}S_0^{-1} + O((S_0)^{-2}), \end{aligned} \tag{6.6}$$

where $I_n(x)$ is the modified Bessel function of the first kind, and I have used its asymptotic series expansion for large x:

$$I_1(x) = \frac{1}{\sqrt{2\pi x}} e^x \left[1 - \frac{3}{8}x^{-1} + O(x^{-2})\right]. \tag{6.7}$$

The (h.o.t)'s in (6.5) and (6.6) would represent the corrections from $\int_C d\tau \ \exp[\sum_{n\geq 5}^{\infty}\eta^{(n)}(\tau - \tau_*)^n/n!]$ and its logarithm, respectively. Here, note that, similarly to (6.3) case, (6.6) can be expressed as '$\ln(\sqrt{x}e^x I_1(x)/\sqrt{2\Delta_{\rm eff}})$ + (h.o.t.)' such as the infinite series of (6.6) may be convergent, though (6.7) might not be in general.[13]

Now, if I consider the CFT at the horizon, (6.6) becomes, using $c_{\rm eff}, \Delta_{\rm eff}$ of (4.14),

$$S_{CFT} = S_{BH} - \frac{1}{2}\ln S_{BH} - \ln(\kappa P) - \frac{3}{8}S_{BH}^{-1} + \frac{1}{2}\ln(8\pi^3) + O(S_{BH}^{-2}). \tag{6.8}$$

Now then, I am ready to test the consistency of our horizon holography by comparing the additional correction terms in (6.2) and (6.8) with P determined at the logarithmic order. But, it is not difficult to show that there is numerical mismatches with that P, such as the period P also get the perturbative corrections in order that the holographic principle holds even for higher orders. To see this, I first note that

$$\begin{aligned} \frac{B_4}{(B_2)^2} &= -24(15J^8l^8 + 170J^6l^6r_+^4 + 1000J^4l^4r_+^8 + 480J^2l^2r_+^{12} + 128r_+^{16}) \\ &\quad \times[(J^2l^2 - 4r_+^4)(3J^2l^2 + 4r_+^4)^3]^{-1} \times S_{BH}^{-1} \\ &\approx 12S_{BH}^{-1} \qquad\qquad ({\rm BTZ}), \end{aligned} \tag{6.9}$$

[13] But note that, in contrast to (6.4) case, $x < 0$ (i.e., $S_0 < 0$) is not allowed in this case, in order that the saddle point approximation of section 3 works here.

$$
\begin{aligned}
\frac{B_4}{(B_2)^2} &= -[((-3+d)l^2+(-1+d)r_+^2)^5(3-d+(-1+d)r_+^2l^{-2})^2((-5+d)(-4+d)(-3+d)^4l^8 \\
&\quad +2(-3+d)^3(-1+d)(-40+11d)l^6r_+^2-2(-3+d)^2(-1+d)^2(-50+(-4+d)d)l^4r_+^4 \\
&\quad -2(-3+d)(-1+d)^3(-4+11d)l^2r_+^6+(-1+d)^4d(1+d)r_+^8)) \\
&\quad \times[(-2+d)l^8((-3+d)l^2-(-1+d)r_+^2)^5(-3+d+(-1+d)r_+^2l^{-2})^6]^{-1}\times S_{BH}^{-1} \\
&\approx \frac{d(d+1)}{d-2}S_{BH}^{-1} \qquad \text{(Schwarzschild-AdS)}\ , \qquad (6.10)
\end{aligned}
$$

such as the bulk entropy (6.2) becomes

$$
\begin{aligned}
S &= S_{BH}-\frac{3}{2}\ln S_{BH}+\frac{12}{8}S_{BH}^{-1}+\frac{1}{2}\ln[32\pi^3l^4(\delta E)^2/(8G)^2]+O(S_{BH}^{-2}) \qquad \text{(BTZ)}, \quad (6.11) \\
S &= S_{BH}-\frac{d}{2(d-2)}\ln S_{BH}+\frac{d(d+1)}{8(d-2)}S_{BH}^{-1}+\frac{1}{2}\ln\left[\frac{8\pi l^4(\delta E)^2}{(d-2)(d-1)^2}\left(\frac{\Omega_{d-2}}{4G}\right)^{\frac{2}{d-2}}\right] \\
&+ O(S_{BH}^{-2}) \qquad \text{(Schwarzschild-AdS)} \qquad (6.12)
\end{aligned}
$$

for the BTZ and Schwarzschild-AdS black holes ($d \geq 4$), respectively. Now then, by comparing (6.11) and (6.12) with (6.8), one finds easily that they match when we re-normalize the period as

$$
\begin{aligned}
P_{ren} &= \frac{8\pi}{\delta E}\left[1-\frac{15}{8}S_{BH}^{-1}+O(S_{BH}^{-2})\right] \qquad \text{(BTZ)}, \qquad (6.13) \\
P_{ren} &= \frac{\sqrt{d-2}\ 8\pi}{\delta E}\left[1-\frac{d^2+4d-6}{8(d-2)}S_{BH}^{-1}+O(S_{BH}^{-2})\right] \qquad \text{(Schwarzschild-AdS)} \quad (6.14)
\end{aligned}
$$

for the BTZ and Schwarzschild-AdS black holes ($d \geq 4$), respectively.

7 Comparison with Other Approaches

a Wess-Zumino-Witten(WZW) Model Approaches at the Horizon

a.1 Lorenzian approach:

For the BTZ black hole, there is alternative derivations of black hole entropy from CFT at the horizon, whose details depend on the signature of the metric, i.e., Lorenzian or Euclidean. I first consider the Lorenzian approach in this section. In this case the CFT comes from an induced $SL(2,\mathbf{R})\times SL(2,\mathbf{R})$ WZW model at the black hole horizon, in the context of Chern-Simons formulation [52] of three-dimensional (Lorenzian) gravity, and the BH entropy comes from the direct counting of its number of states in the large $k=l/4G$ limit [8]. There is an important *qualilative* difference between this and Ref. [12] or Ref. [18], which was analyzed in the previous sections: The BH entropy S_{BH} is a purely "quantum" effect, in contrast to

Ref. [12] or Ref. [18], where the *classical* effect was dominant; but, one can not distinguish between them at the leading order. However, quite interestingly, it is known [40] that *this WZW approaches gives the correct '$-\frac{3}{2}lnS_{BH}$' term already*, as is consistent with the (horizon) holographic principle, though this approach has some undesirable features of the non-unitary Hilbert space and infinite degeneracy of the vacuum [8]. So, the interesting question is then whether this gives the higher order correction terms *correctly* as well, such as this may be considered as the correct CFT candidate for a new AdS/CFT in the context of a horizon holography. But, unfortunately this is not the case, and the WZW model does not provide a correct holography beyond the logarithmic order.

To see this, I first note that the associated partition function becomes

$$\hat{Z}_e[\tau] = \sum_N \rho(N) \exp\left[2\pi i\tau \left(N - \frac{k^2 r_+^2}{l^2}\right)\right] (-i\tau)^{-1/2} \tag{7.1}$$

with the density of states

$$\rho(N) = \sqrt{\frac{2k^2-1}{4k^2}} \sum_{n=0}^{N} \rho_0(N)\rho_0(N-n), \tag{7.2}$$

where ρ_0 is the density of states for an $SL(2,\mathbf{R})$ WZW model [40]. [Here, I am considering the modular *non*-invariant partition function $\hat{Z}_e[\tau]$, following Carlip. But, I have introduced $(-i\tau)^{-1/2}$ factor in (7.1) as well as $(2k^2-1)/(4k^2)$ in (7.2), which have been neglected in [40], to take into account the integral for zero modes: $\int d\bar{\omega} e^{2\pi i\tau(\Delta^+ + \Delta^-)} = \sqrt{\frac{2k^2-1}{-i\tau 4k^2}} e^{-2\pi i\tau k^2 r_+^2/l^2}$; actually $(-i\tau)^{-1/2}$ is just what is needed to incorporate the extra factor in the transformation (A.2) of Ref. [40] into the our standard form (3.3).]

In the large k limit, the three oscillators of $SL(2,\mathbf{R})$ can be treated independently, and (6.5) gives

$$\rho(N) \approx \frac{1}{8} \sum_{n=0}^{N} e^{\sqrt{2}\pi(\sqrt{n}+\sqrt{N-n})} n^{-3/4}(N-n)^{-3/4} \left(1 - \frac{3}{8\sqrt{2}\pi}\frac{1}{\sqrt{n}}\right) \times \left(1 - \frac{3}{8\sqrt{2}\pi}\frac{1}{\sqrt{N-n}}\right) \tag{7.3}$$

up to $O(k^{-2})$ terms. Here, the last two terms came from the second-order correction terms in (6.5).

Furthermore, in the large N limit, the sum in $\rho(N)$ may be approximated as an integral

$$\rho(N) \approx \frac{1}{8} \int_0^\infty dx\, e^{\eta(x)} x^{-3/4}(N-x)^{-3/4} \left(1 - \frac{3}{8\sqrt{2}\pi}\frac{1}{\sqrt{x}}\right) \times \left(1 - \frac{3}{8\sqrt{2}\pi}\frac{1}{\sqrt{N-x}}\right), \tag{7.4}$$

where

$$\eta(x) = \sqrt{2}\pi(\sqrt{x} + \sqrt{N-x}). \tag{7.5}$$

Then, from the steepest descent method, similarly to the previous sections, (7.4) becomes

$$\begin{aligned}
\rho(N) &\approx \frac{2^{3/4}}{8} x_*^{-3/4}(N-x_*)^{-3/4} e^{\eta(x_*)} \times \left(1 - \frac{3}{8\sqrt{2}\pi}\frac{1}{\sqrt{x_*}}\right) \times \left(1 - \frac{3}{8\sqrt{2}\pi}\frac{1}{\sqrt{N-x_*}}\right) \\
&\times \int_0^\infty dx \, \exp\left\{\frac{1}{2}\eta^{(2)}(x-x_*)^2 + \sum_{n\geq 3}^{\infty} \frac{1}{n!}\eta^{(n)}(x-x_*)^n\right\} \\
&= \frac{1}{\sqrt{2\sqrt{2}}} N^{-3/4} e^{2\pi\sqrt{N}} \times \left[1 - \frac{15+12\sqrt{2}}{16\pi} N^{-1/2} + O(N^{-1})\right], \qquad (7.6)
\end{aligned}$$

where

$$\eta(x_*) = 2\pi\sqrt{N} \qquad (7.7)$$

with $x_* = N/2$; here, interestingly, $\eta^{(4)} < 0$, in contrast to all other previous examples. Then, the CFT entropy at the horizon becomes

$$S = \ln\,\rho \approx 2\pi\sqrt{N} - \frac{3}{2}\ln\sqrt{N} - \frac{15+12\sqrt{2}}{16\pi} N^{-1/2} + O(N^{-1}). \qquad (7.8)$$

Now, since $\hat{Z}_e[\tau]$ of (7.1) is dominated by the state with

$$N = \frac{k^2 r_+^2}{l^2} = \frac{S_{BH}^2}{4\pi^2}, \qquad (7.9)$$

which has been known as the physical state condition also [8], the dominant terms in the entropy are

$$S \approx S_{BH} - \frac{3}{2}\ln S_{BH} - \frac{15+12\sqrt{2}}{8} S_{BH}^{-1} + \frac{1}{2}\ln(8\pi^3) + O(S_{BH}^{-2}), \qquad (7.10)$$

which disagrees with the bulk entropy (6.12) at the S_{BH}^{-1} order by the factor of "$-(5+4\sqrt{2})/4$", as well as at the constant term.

Note that although the corrections due to the difference between the sum (7.3) and the integral (7.4), when N is away from the infinity (but, k is kept infinity still), might affect the above result also, but one can check that this is not the case since the effects are only the order of $O(S_{BH}^{-3/2})$ from the Euler-Maclaurin integral formula. (Details are omitted here.)

Hence, one finds that the WZW model approach for the three-dimensional Lorenzian gravity does not favor the (horizon) holography beyond the logarithmic order. This might imply the breakdown of the equivalence between three-dimensional gravity and the corresponding Chern-Simons formulation beyond the logarithmic order since the WZW model depends crucially on the Chern-Simons formulation; but, since several other ingredients which are not directly related to the formulation itself are also involved in this approach, this is not clear at present.

a.2 Euclidean approach:

As a complementary to the previous subsection, it would be also interesting to compare it with the Euclidean continuation of the black hole, which uses a quite different counting of states in terms of the better-understood $SL(2,\mathbf{C})$ WZW model than the poorly understood $SL(2,\mathbf{R})\times SL(2,\mathbf{R})$ WZW model for the Lorenzian black holes [9]. It has been shown that these two different approaches are actually related by a functional Fourier transformation for the different boundary data at the horizons, and they both give the BH entropy at the leading order [27]. The test of the equivalence beyond the leading order is the subject of this subsection.

To this end, I note that, in the large $k=-l/4G$ limit, in which G is analytically continued to a negative value [27, 53], the canonical partition function for the $SL(2,\mathbf{C})$ WZW model on the two-torus with modulus $\tau=\tau_1+i\tau_2$

$$Z_{SL(2,\mathbf{C})}=|Z_{SU(2)}[\tilde{A}]|^2\approx\left|\exp\left[\frac{\pi k}{4\tau_2}\bar{u}^2\right]\bar{\chi}_{0k}(\bar{\tau},\bar{u})\right|^2 \tag{7.11}$$

up to $O(k^{-1})$ terms, where $\tilde{A}_z$ on the horizon is fixed to a constant value [$T_a=-i\sigma_2/2;\ \sigma_a$ are Pauli's matrices]

$$a=-\frac{\pi i}{\tau_2}uT_3\ , \tag{7.12}$$

and χ_{nk} are the Weyl-Kac characters for affine $SU(2)$ [54], which behaves asymptotically for large τ_2

$$\chi_{nk}(\tau,u)\approx\exp\left\{\frac{\pi i}{2}\left[\frac{(n+1)^2}{k+2}-\frac{1}{2}\right]\tau\right\}\frac{\sin\pi(n+1)u}{\sin\pi u}. \tag{7.13}$$

Then the usual number of states, following (3.3), for the states in which the Virasoro generators L_0 and $\bar{L}_0$ have eigenvalues N and $\bar{N}$, respectively, becomes

$$\rho(N,\bar{N})=-\frac{1}{4\pi^2}\int\frac{dq_1}{q_1^{N-\bar{N}+1}}\frac{dq_2}{q_2^{N+\bar{N}-c/12+1}}Z_{SL(2,\mathbf{C})}(\tau), \tag{7.14}$$

where $q_1=e^{2\pi i\tau_1}$, $q_2=e^{-2\pi\tau_2}$. Now, from the physical state condition of $L_0-c/24=\bar{L}_0-c/24=0$ [14] at the horizon, the number of states at the horizon is given by

$$\rho\left(\frac{c}{24},\frac{c}{24}\right)=i\int_{-\infty}^{\infty}d\tau_1\int_{-\infty}^{\infty}d\tau_2\ \exp\left\{\frac{\pi k}{2}\left(\frac{\Theta\tau_1}{2\pi}-\frac{|r_-|}{l}\right)^2-\frac{\pi k}{2}\left[\frac{1}{\tau_2}\left(\frac{\Theta\tau_2}{2\pi}+\frac{r_+}{l}\right)^2-\frac{2\tau_2}{k+2}\right]\right\} \tag{7.15}$$

[14] In the original work [9], $Z_{SL(2,\mathbf{C})}$ was *implicitly* assumed to be $Tr\{e^{2\pi i\tau L_0}e^{-2\pi i\bar{\tau}\bar{L}_0}\}$, which is not modular invariant. However, as can be easily checked, this is not correct since (7.11) is modular invariant [55] as in our starting partition function $Z[\tau,\bar{\tau}]$ of (3.1). Hence, the original work should be understood with a shift $L_0,\bar{L}_0\rightarrow L_0-c/24,\bar{L}_0-c/24$, respectively. Then, the final result is consistent and the same as Carlip's eventually.

This integral, after τ_1 integration, can be evaluated by the steepest descent method with the result

$$\rho\left(\frac{c}{24},\frac{c}{24}\right) \approx e^{\eta(\tau_{2*})}\sqrt{\frac{2\tau_{2*}}{k}}\frac{2\pi}{\Theta}\int_{-\infty}^{\infty} d\tau_2 \exp\left\{\frac{1}{2}\eta^{(2)}(\tau_2-\tau_{2*})^2+\sum_{n\geq 3}^{\infty}\frac{1}{n!}\eta^{(n)}(\tau_2-\tau_{2*})^n\right\}, \quad (7.16)$$

where

$$\eta(\tau_2) = -\frac{\pi k}{2}\left[\frac{1}{\tau_2}\left(\frac{\Theta\tau_2}{2\pi}+\frac{|r_+|}{l}\right)^2\right]+O(k^0), \quad (7.17)$$

which dominates the prefactor $\sqrt{2\tau_{2*}/k}$, gets the maximum

$$\eta(\tau_{2*}) \;=\; \frac{2\pi r_+}{4G}\frac{\Theta}{2\pi}+O(k^0) \quad (7.18)$$

with

$$\tau_{2*} \;=\; \frac{r_+}{l}\frac{\Theta}{2\pi}+O(k^{-1}) \quad (7.19)$$

when $r_+ \gg l$ is considered. Then, entropy $S = \ln\rho(c/24, c/24)$ is computed as [$2\pi-\Theta$ is the deficit angle of the conical singularity at the horizon, and Θ is taken to be 2π on-shell here.]

$$S = S_{BH} + \frac{1}{2}\ln S_{BH} + \frac{1}{2}\ln\left(\frac{-4G}{\pi k l}\right) + (\text{h.o.t.}). \quad (7.20)$$

This show the disagreement, even at the logarithmic order, with the bulk result (4.9), and also shows its *in*equivalence with the Lorenzian approach of the previous subsection beyond the leading order. Moreover, it is unclear, in this result, whether or not the Bekenstein bound, i.e., $S < S_{BH}$ [56] is satisfied since both S_{BH} and k are extremely large.

Recently, it has been suggested to consider the modular invariant density of states [55] by employing the modular invariant measure $d\tau_1 d\tau_2/(\tau_2)^2$ instead of the usual measure $d\tau_1 d\tau_2$ in (7.15), and in that construction the correct logarithmic term '$-(3/2)\ln S_{BH}$', which agrees with the bulk result (4.9), was obtained. But, its relevance to the usual thermodynamics is not clear yet, which will be more discussed in the final section. Moreover, even with that construction, the entropy disagrees with the bulk result (6.11) again beyond the logarithmic order. (See Appendix **A** for details.)

b Quantum Geometry Approach of the Horizon

For four-dimensional non-rotating black holes with or without a cosmological constant or electric, magnetic, and dilatonic charges, there is an alternative derivation of black hole entropy

from the quantum geometry approach of Ashtekar et al. that counts the boundary states of a three-dimensional Chern-Simons theory at the horizon [10, 57, 41]. There were some confusions about the logarithmic-correction term in this approach, by identifying the entropy in this approach with the usual (micro-canonical) entropy $S = \ln\Omega(E)$ and generalizing this result in $d = 4$ to other dimensions [40]; as a result of this, a wrong behavior of central charge c (or P) in the Cardy's formula (4.16) has been speculated in Ref. [40], as remarked in section 4. Clarification of this issue is considered in this subsection, with the help of a recent observation by Chatterjee and Majumdar [58].

To this end, I start by noting that the partition function in quantum geometry approach is represented by

$$Z[\hat{\beta}] = \sum_{n=0}^{p} \mathcal{N}(E(A_n))e^{-\hat{\beta}E(A_n)}, \tag{7.21}$$

where A_n is the area eigenvalue of the horizon with n punctures each of which carries a spin $1/2$, and $\mathcal{N}(E(A_n))$ is the degeneracy of the energy level $E(A_n)$ [10, 41]. In the very large p limit, this may be approximated as an integral

$$Z[\hat{\beta}] = \int_0^\infty dx\ \mathcal{N}(E(A(x)))e^{-\hat{\beta}E(A(x))}. \tag{7.22}$$

Changing integration variable from x to E yields

$$Z[\hat{\beta}] = \int_0^\infty dE \left|\frac{dE}{dx}\right|^{-1} \mathcal{N}(E)e^{-\hat{\beta}E} \tag{7.23}$$

such as, by comparing with the usual form of (2.4), one obtains

$$\Omega(E) = \mathcal{N}(E)\left|\frac{dE}{dx}\right|^{-1} \delta E. \tag{7.24}$$

Then, the usual micro-canonical entropy $S = \ln\Omega(E)$ becomes [58]

$$S = S_{QG} - \ln\frac{|dE/dx|}{\delta E}, \tag{7.25}$$

where

$$S_{QG} = \ln\mathcal{N}(E(p)), \tag{7.26}$$

which is the entropy defined in quantum geometry approach [10, 57, 41]. As we shall see, the additional term in (7.25) is the key ingredient which resolves the above mentioned confusions.

Now, from counting the number of conformal blocks of a two-dimensional $SU(2)_{\hat{k}}$ WZW model with the level $\hat{k} = A/(8\pi\gamma G)$ that lives on the punctured 2-sphere at $\hat{k} \to \infty$ limit, one finds

$$\mathcal{N}(E(p_0)) \approx \begin{pmatrix} p_0 \\ p_0/2 \end{pmatrix} - \begin{pmatrix} p_0 \\ (p_0/2-1) \end{pmatrix} \tag{7.27}$$

for the largest number of punctures p_0 with all spins $j_n = 1/2$, which is given by

$$p_0 = \frac{A}{4G}\frac{\gamma_0}{\gamma}, \tag{7.28}$$

where $\gamma_0 = 1/(\pi\sqrt{3})$, and γ is the Barbero-Immirzi parameter [59]. Then, for large p_0, one finds, from the asymptotic expansion formula of the factorial function [15],

$$\begin{aligned} S_{QG} &\approx p_0 \ln 2 - \frac{3}{2}\ln p_0 - \frac{9}{4}p_0^{-1} + \frac{1}{2}\ln(8/\pi) + O(p_0^{-2}) \\ &= S_{BH} - \frac{3}{2}\ln S_{BH} - \frac{9}{4}\ln 2 S_{BH}^{-1} + \frac{1}{2}\ln[(\ln 2)^3 8/\pi] + O(S_{BH}^{-2}) \end{aligned} \tag{7.29}$$

by choosing $\gamma = \gamma_0 \ln 2$ [10, 41][16]. Note that this result applies to the four-dimensional Schwarzschild-AdS as well as the Schwarzschild black holes since the formulation depends crucially on the local properties of the horizon [57]. However, when we compute the usual micro-canonical entropy $S = \ln\Omega(E)$, there is a sharp difference due to the additional term of (7.25).

To see this, let us first consider the four-dimensional Schwarzschild-AdS black hole, in which the black hole mass M is given by, from (5.3),

$$2GM = r_+\left(1 + \frac{r_+^2}{l^2}\right). \tag{7.30}$$

For a very large black hole in the presence of a cosmological constant $\Lambda = -3/l^2$, i.e., $r_+ \gg l$, this reduces to

$$\begin{aligned} M &\approx \frac{1}{2Gl^2}r_+^3 \\ &= \frac{1}{2Gl^2(4\pi)^{3/2}}A^{3/2}, \end{aligned} \tag{7.31}$$

where I have used $A = 4\pi r_+^2$ for the horizon area. With this mass-area relation one finds

$$\frac{dE}{dx} = \frac{dE}{dA}\frac{dA}{dx} = \frac{3}{2(4\pi)^{3/2}\sqrt{G}l^2}S_{BH}^{1/2}\frac{dA}{dx}. \tag{7.32}$$

[15]Note that the p_0^{-1} and the constant terms differ from Ref. [41].

[16]In Ref.[40], S_{QG} has been identified implicitly in arbitrary dimensions as the micro-canonical entropy S. But, as we have seen in (7.25), this is not correct because of the second term, which can not be neglected. Furthermore, in higher dimensions, the applicability of the formulation has not been proved yet.

Now, by plugging $dA/dx \approx dA/dp_0 = 4G\gamma/\gamma_0$, which is a constant factor independent of the area A, one obtains finally the microscopic entropy $S = \ln\Omega(E)$, from (7.25) and (7.29),

$$S \approx S_{BH} + \left(-\frac{3}{2} - \frac{1}{2}\right)\ln S_{BH} - \frac{9}{4}\ln 2\ S_{BH}^{-1} + \frac{1}{2}\ln\frac{128\pi^2(\ln 2)l^4(\delta E)^2}{9G} + O(S_{BH}^{-2}). \qquad (7.33)$$

This disagrees with the bulk result (5.13), even at the logarithmic order. And, as was noted in Ref.[41] and explicitly shown in Ref. [60], the constant and the $O(S_{BH}^{-1})$ terms might be affected by taking the level $\hat{k}$ away from the asymptotic value (∞), which we have assigned above in the integral (7.21), or by including the spin values higher than 1/2 such as we are away from the largest number of punctures (7.28), but the logarithmic term is not affected.

On the other hand, in the absence of a cosmological constant, i.e., Schwarzschild black hole, (7.30) reduces, for *any* black hole size, to

$$M = \frac{r_+}{2G} = \frac{1}{4\sqrt{\pi}G}A^{1/2} \qquad (7.34)$$

such as one finds, instead,

$$S \approx S_{BH} + \left(-\frac{3}{2} + \frac{1}{2}\right)\ln S_{BH} - \frac{9}{4}\ln 2\ S_{BH}^{-1} + \frac{1}{2}\ln[128\ln 2\ G(\delta E)^2] + O(S_{BH}^{-2}). \qquad (7.35)$$

But, unfortunately, there is no comparable bulk entropy computation in this case since the canonical partition function (2.4) diverges such as the expansion of $\Omega(E)$ as in (2.6) is not defined, though, interestingly, this has the same logarithmic term of (5.8) for $d = 4$: We need an independent framework to compute $\Omega(E)$; a possible way would be to consider the canonical ensemble of the black hole within a finite cavity [61], though it is too artificial [31], but because of the finite size of the cavity compared to the Schwarzschild radius $r_+ = 2M$, the result can not be expanded as in (7.35) [62]; we need an estimation in the micro-canonical ensemble from first principles. In this circumstance, it would be interesting to compare with the horizon-CFT approach, which applies also to the Schwarzschild black hole. In this approach, one obtains the microscopic entropy, from (4.16) and (6.6), as

$$S_{CFT} = S_{BH} - \ln P - \frac{3}{8}S_{BH}^{-1} + \frac{1}{2}\ln(32G) + O(S_{BH}^{-2}), \qquad (7.36)$$

where I have used

$$\kappa = \frac{\pi}{4GM} = \frac{\pi^{3/2}}{2\sqrt{G}}S_{BH}^{-1/2}. \qquad (7.37)$$

Note that "$-(1/2)\ln S_{BH}$" term cancels '$-\ln\kappa \sim (1/2)\ln S_{BH}$' in (4.16), such as only "$-\ln P$" term remains in (7.36). The quantum geometry result (7.35) would agree with this result if I choose

$$P = aS_{BH} + b \qquad (7.38)$$

with the appropriate constants a and b; $a^2 = 1/(4\ln2(\delta E)^2), b = ((9/4)\ln2 - (3/8))a$. But, this would be quite questionable since there is the disagreement in the Schwarzschild-AdS case always, as already was noted in (7.33), and there is no resolution in the present context yet; a modification in the classical action of Ref.[10] might be needed for the resolution, but it not clear at present.

In summary for quantum geometry approach, there is the disagreement beyond the logarithmic term with the bulk result of Schwarzschild-AdS black holes, but there is no way to test the holography for the Schwarzschild black hole case, i.e., Flat/CFT due to the absence of its corresponding bulk entropy computation.

8 Conclusion and Open Questions

I have tested the holographic principle for mainly AdS spaces by examining the logarithmic and higher order corrections to the Bekenstein-Hawking entropy of black holes. For the BTZ black hole, the holographic correspondence with the screen at spatial infinity has some disagreement beyond the logarithmic correction (by the factor of 2 at the logarithmic order), which may be in contrast to the AdS/CFT predictions. On the other hand, a holography at the event horizon from Carlip's horizon-CFT approach of black hole entropy in any dimension [12, 13, 14] is more flexible, and can be made to be satisfied by choosing an appropriate period parameter P, which has been arbitrary for the leading-order computation. The analysis on the higher dimensional Schwarzschild black holes in AdS space shows a universality of the choice P, except for a dimension-dependent factor. I have also compared with several other approaches, i.e., the induced WZW model approaches at the horizon for the Lorenzian and the Euclidean BTZ black holes, and also the quantum geometry approach of Ashtekar et al. at the horizon for the four-dimensional Schwarzschild and Schwarzschild-AdS black holes. In these analysis I have shown that none of these other approaches satisfies the (horizon) holographic principle, in contrast to our horizon-CFT approach. As a byproduct, I have clarified some confusions in the quantum geometry approach which result from some misunderstandings of the definition of entropy. However, there remain several open questions, which are listed below.

1. *Quantum corrections* [17]: While I have used the expansion (2.6) around $\beta' = 0$, I have not touched $Z[\beta]$, which may include all quantum loop corrections by $Z[\beta] = Tr(exp(-\beta H)) = \int d[g] exp(iI[g])$ [23]. If we consider the loop corrections of the gravitational fields we need to generalize the Einstein gravity to a higher curvature gravity [63]–in this case, the canonical entropy (2.9) is not simply the BH form (2.10), but also includes a sum of curvature invariants

[17] I thank for S. Deser, D. V. Fursaev, J. Maldacena, and S. N. Solodukhin for discussion.

integrated over the horizon [64] [18]– with a *possible* UV-finite lnA correction. But this would not affect *much* our result for large black holes from the following reasons:

i). In this paper, I have considered mainly the black holes in AdS space, which is stable under the fluctuations of thermal gravitons [31]. In this case, it seems that the usual lnA correction from some ambiguity in the renormalization for the d=4 Schwarzschild black hole, due to lack of a natural length scale μ other than $r_+ = 2Ml^2_{Planck}$ [66, 32, 35], would not appear [67] due to another natural length scale $l \sim \sqrt{-\Lambda}$ in AdS. All the other terms are UV-divergent which can be renormalized away.

ii). Now, involving the higher curvature terms in entropy, which is renormalized, these would be very small for large black holes, as in this paper, since the curvature near the horizon decreases as the horizon area A increases. (For more details, see the argument by Carlip in Appendix B of [40].) Moreover, for three dimensions, in particular, one can expect no gravitational loop correction due to the absence of propagating graviton [68], such as our results for the BTZ black hole in section 4 are not affected by quantum corrections, except for some possible finite terms which would be renormalized away [9]. However, an explicit computation of those higher curvature terms in the entropy for the CFT as well as the bulk would be quite interesting.

2. *De-Sitter space*: From the empirical reason, holography for de-Sitter space would be more interesting, but the situation is not so affirmative as that of AdS space: The only thing I can show (the details will be appeared elsewhere [28]) is that, by generalizing the steepest descent method for "imaginary" Δ and c , the KdS_3 solution [3] has the same factor of 2 discrepancy at the logarithmic order for the entropy at spatial infinity as the BTZ black hole; this shows clearly the usual $\mathrm{dS}_3/\mathrm{CFT}_2$ *might* not be correct either beyond the logarithmic correction. But, when we consider the entropy at the cosmological horizon, which is the only horizon in this case, the steepest descent approximation is questionable since the convergence of an expansion around the steepest descent path is not guaranteed, which is essentially due to finite r_+/l; this problem is similar to the situation for the black holes in a finite cavity [61]. We need other ways to compute the entropy corrections such that one can get a convergent expansion around the steepest descent path with some controlling parameters.

3. *Flat space*: Can we "directly" compute the bulk density of states $\Omega(E)$ for the asymptotically flat Schwarzschild black holes, without recourse to the Laplace transformation of the canonical partition function (2.2), which is divergent in this case ? This can be used for testing F(lat)/CFT correspondence, which has been recently considered [4, 6].

4. *Modular invariance*: In the CFT side of section 3, the modular invariance–especially

[18] The higher curvature effect in the black hole entropy would be also reproduced by the Cardy's formula with the corresponding corrections in the central charge and L_0 eigenvalue. See [65] for this possibility.

the invariance under $\tau \to -1/\tau$–of the partition function $Z[\tau, \bar{\tau}]$ in (3.1) has been a crucial role in determining its density of states $\rho(\Delta, \bar{\Delta})$ of (3.5) perturbatively. But, this property is clearly absent in the usual canonical partition function in section 2, which implies some mismatches between $\Omega(E)$ in (2.2) and $\rho(\Delta, \bar{\Delta})$ in (3.3). One might consider the modular invariant measure as in [55, 69] in order *not* to double-count the states which are connected by the modular transformations. But, in this case the usual connection of $\tau \sim i\beta/l$ [70] would be lost for large β (i.e., small temperature T) case. A possible resolution might be to generalize the usual thermodynamics relations in section 2 such that a similar duality in the thermal temperature T is favored as in Ref.[71]. Even with this unclear circumstance in the large β, our main computation for small β (i.e., high T) black holes seems to be quite safe still. But, for an affirmative answer, we need to compute directly the density of states $\rho(\Delta, \bar{\Delta})$ without recourse to the modular invariance. Is there any way to do this ?

5. *String theory*: In string theory side, similar disagreements in two-loop tests of the PP-wave/YM correspondence have been reported [72, 73], and there has been no clear resolution yet. The disagreement are the factor of 2 for Ref.[72], and approximately 2 (i.e., 19/8) for Ref. [73]; for the latter case, a resolution was proposed in the later paper [73], but its independent confirmation seems to be unclear so far. This factor of 2 disagreement resembles the same factor disagreement for the BTZ black hole entropy at the logarithmic order. Is this just an accidental coincidence, or is there any deeper reason ?

6. *Much higher order corrections*: In the evaluation of much higher order corrections ($n > 4$) for $\rho(\Delta, \bar{\Delta})$ we can use the exact formula of Rademacher to any order, which implies the integral with an infinite sum in the exponent of the integrand in (3.6) can be computed. Then, can we obtain, from the Rademacher's exact formula, the integral formula for $\int_{-\infty}^{\infty} dx \; exp\{\sum_{n=2}^{N} a_n x^n\}$ for arbitrary $N > 4$ such that the integral in $\Omega(E)$ of (2.6) can be computed to arbitrary orders also ? This would be an interesting problem in mathematics itself also.

7. *Chern-Simons theory versus gravity*: It is well known that the three-dimensional Chern-Simons action with an appropriate gauge group is equivalent to the three-dimensional pure gravity action "on-shell" [52], and this is true even for the usual boundary term "$\oint 2K$" at spatial infinity as well as the bulk Einstein-Hilbert action [74]. Then, does our result in section 7.a imply that this equivalence is not true at quantum level, which is "off-shell" necessarily, or the boundary term of the Chern-Simons action for the horizon should be modified in accordance with the appropriate horizon term in gravity side [14] ?

Note added: While this paper was being written, a paper [75] appeared which noted also the factor of 2 mismatch for the BTZ black hole between the bulk and Strominger's logarithmic correction to the BH entropy. But there, in contrast to my paper, this mismatch was neglected and not seriously considered .

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Appendix A

In this appendix, I compute the black hole entropy of Euclidean black holes of section 7 a.2 for the modular invariant density of states, as suggested by Govindarjan et al. [55]. The suggested form of the density of states corresponds to a modified density of states, in contrast to the usual $\rho(c/24, c/24)$ of (7.15),

$$\tilde{\rho}\left(\frac{c}{24}, \frac{c}{24}\right) = i\int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 \frac{1}{(\tau_2)^2} Z_{SL(2,\mathbf{C})}(\tau) \tag{A.1}$$

with the modular invariant measure, by incorporating $1/(\tau_2)^2$ factor, which is well-known factor in string theory context [69]. Here, since $Z_{SL(2,\mathbf{C})}(\tau)$ is modular invariant by itself on a two-torus, the density of states $\tilde{\rho}$ itself is modular invariant also; of course, the density of states for the higher modes $\rho(N, \bar{N})$ is not modular invariant still, even with the $1/(\tau_2)^2$ factor, and needs a more generalized factor in that case.

The integral (A.1) can be evaluated by the steepest descent method as in (3.6)

$$\begin{aligned} \tilde{\rho}\left(\frac{c}{24}, \frac{c}{24}\right) \approx\ & \frac{e^{\eta(\tau_{2*})}}{(\tau_{2*})^2}\sqrt{\frac{2\tau_{2*}}{k}}\frac{2\pi}{\Theta}\int_{-\infty}^{\infty} d\tau_2 \, \exp\left\{\frac{1}{2}\eta^{(2)}(\tau_2-\tau_{2*})^2 + \sum_{n\geq 3}^{\infty}\frac{1}{n!}\eta^{(n)}(\tau_2-\tau_{2*})^n\right\} \\ & \times\left[1+\sum_{m\geq 1}^{\infty}\frac{1}{m!}(\tau_{2*})^2[(\tau_{2*})^{-2}]^{(m)}|_{\tau_{2*}}(\tau_2-\tau_{2*})^m\right], \end{aligned} \tag{A.2}$$

where $\eta(\tau_2) = -\frac{\pi k}{2}[\frac{1}{\tau_2}(\frac{\Theta\tau_2}{2\pi}+\frac{|r_+|}{l})^2] + O(k^0)$, which dominates the prefactor $(\tau_{2*})^{-2}\sqrt{2\tau_{2*}/k}$, gets the same maximum as (7.18) when $r_+ \gg l$ is considered; there are no terms of odd power in $(\tau_2 - \tau_{2*})$ in the exponent of (A.2), due to the same reason for (3.6) case. Then, from the integral formula (6.3) and the asymptotic form of the modified Bessel function of the second kind, one obtains black hole entropy $\tilde{S} \equiv \ln\tilde{\rho}(\frac{c}{24}, \frac{c}{24})$ as follows:

$$\tilde{S} = S_{BH} - \frac{3}{2}\ln S_{BH} + \frac{24}{8}S_{BH}^{-1} + \frac{1}{2}\ln\left(\frac{\pi^3 l^2}{G^2}\right) + O(S_{BH}^{-2}, k^{-1}). \tag{A.3}$$

Note that one obtains the correct '$-3/2$' factor in the logarithmic term, which agrees with the bulk result (4.9) [55]: This is essentially due to the additional factor $1/(\tau_2)^2$ in the measure, which "reduces" the counted number of states in general ['$-2\ln S_{BH}$' in our case] by removing the over-counted states due to the modular transformation. However, even in this construction, the higher order terms, which are actually the same as in the case without the $1/(\tau_2)^2$ factor, disagrees with the bulk result (6.11) again.

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